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Book Chapter

THE GENERALIZATIONS OF THE MULTIDIMENSIONAL FOURIER TRANSFORM

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The generalizations of the multidimensional Fourier transform

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Abstract

This work, deal a brief introduction of multidimensional $(M-)$ Fourier transform and its generalization. So, we give the definition of such transformation and we establish their properties enriched by several illustrative examples. Additionally, we give the intrinsic properties of the generalized version as: $M-$windowed, $M-$fractional, $M-$linear canonical transform , and $M-$quadratic-phase Fourier transforms.

Some Keywords: Multidimensional Fourier transforms, Windowed Fourier transform, Fractional Fourier transform, Linear canonical transform, Quadratic-phase Fourier transform.

1 Introduction

The harmonic analysis plays a central role in applied mathematics, engineering and signal processing. Its main objective is certainly the frequency domain analysis and/or the spectral analysis which resides in the signal instead of the time-domain approach. The last approach depends on the choice of the relationship equipped with the space. In contrast, the frequency-domain allows us to implement some algorithms for estimating, modeling and forecasting the signal independently on such an order chosen (partial or total) on the space. One, among others, concerns the harmonic analysis is the Fourier transform (FT for short) which is used widely in various areas, it becomes a powerful tool for analyzing linear, no-linear, stationary and non-stationary real data including volatility analysis, financial mathematics and image processing and so on. Particularly, in time series analysis, it helps us to identify some dataset encountered in practice. The fundamental limitation of the unidimensional FT is that, in practice, certain signals are strongly depends on several indexes for instance spatio-temporal, spacial econometrics data, etc... Thus the resort to multidimensional Fourier transform (MFT) is however inevitable. However, the MFT have know some limit or little uses because all properties of the signal are global in scope. Information about local features of the signal becomes a global property of the signal in frequency domain. To remedy these drawbacks of MFT, we introduce its generalization that includes short-time MFT ($S-MFT$) by performing the MFT on a block-by-block basis rather than to process the entire signal at once (see Bahat (2023) for more details). Moreover, different novel generalizations of the classical MFT came into existence via.: the fractional MFT ($F-MFT$), the linear canonical MFT ($L-MFT$), the quadratic-phase MFT ($Q-MFT$), and so on. As a generalization of classical MFT, the $F-MFT$, $L-MFT$ and the $Q-MFT$ gained its ground intermittently and profoundly influenced several branches of science and engineering including signal and image processing, quantum mechanics, neural networks, differential equations, optics, pattern recognition, radar, sonar, and communication systems.

The rest of the paper is organized as follow. In Section 2, we introduced the definition of the multidimensional Fourier transform and some basic results from a mathematical point of view. Section 3, 4 and 5 respectively, studied the multidimensional windowed Fourier transform, multidimensional fractional Fourier transform and multidimensional linear canonical transform and its properties respectively. Finally, we introduced the notion of multidimensional quadratic-phase Fourier transform. Section 7 concludes the article and we end with a summary table.

2 Multidimensional Fourier transform

In the sequel, if $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ two vectors in $\mathbb{R}^n$, we shall note $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}.\mathbf{b} = \sum_{i=1}^{n} a_i b_i$, $\frac{\mathbf{a}}{\mathbf{b}} = (\frac{a_1}{b_1}, ..., \frac{a_n}{b_n})$ if $b_1, ..., b_n \neq 0$, $\mathbf{a} \leq \mathbf{b}$ means that $a_i \leq b_i, i = 1, ...n$.

Definition 1 *The multidimensional Fourier transform of any multidimensional signal $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$ is defined and denoted for all $\zeta \in \mathbb{R}^n$ as*

$$\mathcal{F}[x(\mathbf{t})](\zeta) = \widehat{x}(\zeta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}} x(\mathbf{t})\, d\mathbf{t} \tag{2.1}$$

and corresponding inversion formula is given by

$$\mathcal{F}^{-1}(\mathcal{F}[x(\mathbf{t})](\zeta))(\mathbf{t}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}} \mathcal{F}[x(\mathbf{t})](\zeta)\, d\zeta. \tag{2.2}$$

$$x(\mathbf{t}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}} \widehat{x}(\zeta)\, d\zeta.$$

Example 2.1 *The most commonly used ones in image processing are the rect function in two dimensions:*

$$rect(x, y) = \begin{cases} 1 & \text{if } |x| \leq 0.5 \text{ and } |y| \leq 0.5; \\ 0 & \text{otherwise.} \end{cases}$$

The $2D$ Fourier transform

$$\begin{aligned} \widehat{x}(u, v) &= \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} rect(x, y)\, e^{-i(u.x+v.y)} dxdy = \frac{1}{(2\pi)} \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} e^{-i(u.x+v.y)} dxdy \\ &= \frac{1}{(2\pi)} \int_{-0.5}^{+0.5} e^{-i(x.u)} dx \int_{-0.5}^{+0.5} e^{-i(v.y)} dy = \frac{\sin \pi u}{u} . \frac{\sin \pi v}{v} \\ &= \sin c(u). \sin c(v) \end{aligned}$$

where $\sin c(.)$ is the sinus cardinal function. (Note that the rect (x, y) is separable, the resulting 2-D Fourier transform is the product of the corresponding $1 - D$ Fourier transforms).

Example 2.2 *Two-dimensional Fourier transform of some functions is given in the following tables*

$x(t_1,t_2)$	$\mathcal{F}[x(t_1,t_2)](\zeta_1,\zeta_2)$
$redt(t_1,t_2)$	$\frac{\sin(\pi\zeta_1)}{\pi\zeta_1}\frac{\sin(\pi\zeta_2)}{\pi\zeta_2}$
$d.rect(at_1,bt_2)$	$\frac{d}{ab}\sin c\left(\frac{\zeta_1}{a}\right)\sin c\left(\frac{\zeta_2}{b}\right)$
$\exp\left\{-\frac{t_1^2+t_2^2}{2}\right\}$	$2\pi\exp\left\{-2\pi^2\left(\zeta_1^2+\zeta_2^2\right)\right\}$
$\exp\left\{-2\pi\sqrt{t_1^2+t_2^2}\right\}$	$\frac{1}{2\pi^2}\frac{1}{\left(1+\zeta_1^2+\zeta_2^2\right)^{3/2}}$
$\cos\left(\pi\left(t_1^2+t_2^2\right)\right)$	$\sin\left(\pi\left(\zeta_1^2+\zeta_2^2\right)\right)$
$\exp\left\{i\pi\left(t_1^2+t_2^2\right)\right\}$	$i\exp\left\{-i\pi\left(\zeta_1^2+\zeta_2^2\right)\right\}$
$\delta(t_1,t_2)$	1

The multidimensional Fourier transform has properties that are completely analogous to the familiar properties of the 1D Fourier transform as shown in the following

Theorem 2.1 (Linearity) *Let $x(\mathbf{t})$ and $y(\mathbf{t})$ in $\mathbb{L}^2(\mathbb{R}^n)$, then*

$$\mathcal{F}[Ax(\mathbf{t})+By(\mathbf{t})](\zeta)=A\mathcal{F}[x(\mathbf{t})](\zeta)+B\mathcal{F}[y(\mathbf{t})](\zeta).$$

Proof. Let $s(\mathbf{t})=Ax(\mathbf{t})+By(\mathbf{t})$, then

$$\begin{aligned}\mathcal{F}[s(\mathbf{t})](\zeta)&=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{t}}s(\mathbf{t})\,d\mathbf{t}\\&=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{t}}\left(Ax(\mathbf{t})+By(\mathbf{t})\right)d\mathbf{t}\\&=A\left[\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{t}}x(\mathbf{t})d\mathbf{t}\right]+B\left[\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{t}}y(\mathbf{t})d\mathbf{t}\right]\\&=A\mathcal{F}[x(\mathbf{t})](\zeta)+B\mathcal{F}[y(\mathbf{t})](\zeta).\end{aligned}$$

Theorem 2.2 (Translation) *The multidimensional Fourier transform of any function $x(\mathbf{t}-\mathbf{k})$ is given by*

$$\mathcal{F}[x(\mathbf{t}-\mathbf{k})](\zeta)=e^{-i\mathbf{k}.\zeta}\mathcal{F}[x(\mathbf{t})](\zeta).$$

Proof. From Definition (1), we have

$$\begin{aligned}\mathcal{F}[x(\mathbf{t}-\mathbf{k})](\zeta)&=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{t}}x(\mathbf{t}-\mathbf{k})\,d\mathbf{t}\\&=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.(\mathbf{u}+\mathbf{k})}x(\mathbf{u})\,d\mathbf{u},\ \mathbf{u}=\mathbf{t}-\mathbf{k}\\&=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{k}}e^{-i\zeta.\mathbf{u}}x(\mathbf{u})\,d\mathbf{u}\\&=\frac{1}{\left(\sqrt{2\pi}\right)^n}e^{-i\zeta.\mathbf{k}}\int_{\mathbb{R}^n}e^{-i\zeta.\mathbf{u}}x(\mathbf{u})\,d\mathbf{u}\\&=e^{-i\zeta.\mathbf{k}}\mathcal{F}[x(\mathbf{t})](\zeta).\end{aligned}$$

Theorem 2.3 (Modulation) *The multidimensional Fourier transform of any function* $e^{-i\zeta_0.\mathbf{t}}x(\mathbf{t})$ *is given by*

$$\mathcal{F}\left[e^{-i\zeta_0.\mathbf{t}}x(\mathbf{t})\right](\zeta) = \mathcal{F}\left[x(\mathbf{t})\right](\zeta - \zeta_0).$$

Proof. From Definition, we have

$$\begin{aligned}\mathcal{F}\left[e^{i\zeta_0.\mathbf{t}}x(\mathbf{t})\right](\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\mathbf{t}.\zeta}e^{-i\zeta_0.\mathbf{t}}x(\mathbf{t})d\mathbf{t} \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\mathbf{t}.(\zeta-\zeta_0)}x\left(\mathbf{t}\right)d\mathbf{t}, \\ &= \mathcal{F}\left[x(\mathbf{t})\right](\zeta - \zeta_0).\end{aligned}$$

Theorem 2.4 *The multidimensional Fourier transform of the functions* $x(\mathbf{t})$ *and* $y(\mathbf{t})$ *in* $\mathbb{L}^2(\mathbb{R}^n)$ *satisfies the following orthogonality relation*

$$\left\langle \mathcal{F}\left[x(\mathbf{t})\right], \mathcal{F}\left[y(\mathbf{u})\right]\right\rangle = \left\langle x(\mathbf{t}), y(\mathbf{u})\right\rangle.$$

Proof. We have

$$\begin{aligned}\left\langle \mathcal{F}\left[x(\mathbf{t})\right], \mathcal{F}\left[y(\mathbf{u})\right]\right\rangle &= \int_{\mathbb{R}^n} \mathcal{F}\left[x(\mathbf{t})\right](\zeta)\,\overline{\mathcal{F}\left[y(\mathbf{u})\right](\zeta)}d\zeta \\ &= \int_{\mathbb{R}^n}\left(\frac{1}{\sqrt{(2\pi)^n}}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}x\left(\mathbf{t}\right)d\mathbf{t}\right)\overline{\left(\frac{1}{\sqrt{(2\pi)^n}}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{u}}y\left(\mathbf{u}\right)d\mathbf{u}\right)}d\zeta \\ &= \int_{\mathbb{R}^n}\left(\frac{1}{\sqrt{(2\pi)^n}}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}x\left(\mathbf{t}\right)d\mathbf{t}\right)\left(\frac{1}{\sqrt{(2\pi)^n}}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{u}}\overline{y\left(\mathbf{u}\right)}d\mathbf{u}\right)d\zeta \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\left(\mathbf{t}\right)\overline{y\left(\mathbf{u}\right)}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\zeta.(\mathbf{u}-\mathbf{t})}d\zeta\right)d\mathbf{t}d\mathbf{u} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x\left(\mathbf{t}\right)\overline{y\left(\mathbf{u}\right)}\left(\delta\left(\mathbf{u}-\mathbf{t}\right)\right)d\mathbf{t}d\mathbf{u} \\ &= \int_{\mathbb{R}^n} x\left(\mathbf{t}\right)\overline{y\left(\mathbf{u}\right)}d\mathbf{t} \\ &= \left\langle x(\mathbf{t}), y(\mathbf{u})\right\rangle.\end{aligned}$$

Next, we show that the multidimensional inverse Fourier operator is the adjoint of the multidimensional Fourier operator.

Theorem 2.5 *Let* $x(\mathbf{t})$ *and* $y(\mathbf{t})$ *in* $\mathbb{L}^2(\mathbb{R}^n)$, *then*

$$\left\langle \mathcal{F}\left[x(\mathbf{t})\right](\zeta), y(\zeta)\right\rangle = \left\langle x(\mathbf{t}), \mathcal{F}^{-1}\left[y\right](\mathbf{t})\right\rangle.$$

Proof. We have

$$
\begin{aligned}
\langle \mathcal{F}\left[x(\mathbf{t})\right](\zeta), y(\zeta)\rangle &= \int_{\mathbb{R}^n} \mathcal{F}\left[x(\mathbf{t})\right](\zeta)\,\overline{y(\zeta)}d\zeta \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}x(\mathbf{t})\,d\mathbf{t}\right)\overline{y(\zeta)}d\zeta \\
&= \int_{\mathbb{R}^n} x(\mathbf{t})\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}\overline{y(\zeta)}d\zeta\right)d\mathbf{t} \\
&= \int_{\mathbb{R}^n} x(\mathbf{t})\overline{\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}}y(\zeta)\,d\zeta\right)}d\mathbf{t} \\
&= \int_{\mathbb{R}^n} x(\mathbf{t})\,\overline{\mathcal{F}^{-1}\left[y\right](\mathbf{t})}d\mathbf{t} \\
&= \left\langle x(\mathbf{t}), \mathcal{F}^{-1}\left[y\right](\mathbf{t})\right\rangle.
\end{aligned}
$$

Theorem 2.6 (Convolution) *Let $x(\mathbf{t})$ and $y(\mathbf{t})$ in $\mathbb{L}^2(\mathbb{R}^n)$, then*

$$
\mathcal{F}\left[(x*y)(\mathbf{t})\right](\zeta) = (2\pi)^{\frac{n}{2}}\,\mathcal{F}\left[x(\mathbf{t})\right](\zeta)\,\mathcal{F}\left[y(\mathbf{t})\right](\zeta).
$$

where $x*y$ denotes the convolution of the functions $x(\mathbf{t})$ and $y(\mathbf{t})$ and is given by

$$
(x*y)(\mathbf{u}) = \int_{\mathbb{R}^n} x(\mathbf{t})y(\mathbf{u}-\mathbf{t})d\mathbf{t}.
$$

Proof. By applying definition of multidimensional Fourier transform to the convolution of the functions $x(\mathbf{t})$ and $y(\mathbf{t})$, we obtain

$$
\begin{aligned}
\mathcal{F}\left[(x*y)(\mathbf{t})\right](\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}(x*y)(\mathbf{t})\,e^{-i\zeta.\mathbf{t}}d\mathbf{t} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} x(\mathbf{u})y(\mathbf{t}-\mathbf{u})d\mathbf{u}\right)e^{-i\zeta.\mathbf{u}}d\mathbf{u} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x(\mathbf{u})y(\mathbf{v})e^{-i\zeta.(\mathbf{u}+\mathbf{v})}d\mathbf{v}d\mathbf{u}, \text{ where } \mathbf{v}=\mathbf{t}-\mathbf{u} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{u}}x(\mathbf{u})y(\mathbf{v})e^{-i\zeta.\mathbf{v}}d\mathbf{v}d\mathbf{u} \\
&= \left(\sqrt{2\pi}\right)^n\left\{\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{u}}x(\mathbf{u})d\mathbf{u}\right\}\left\{\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{v}}y(\mathbf{v})d\mathbf{v}\right\} \\
&= \left(\sqrt{2\pi}\right)^n\mathcal{F}\left[x(\mathbf{t})\right](\zeta)\,\mathcal{F}\left[y(\mathbf{t})\right](\zeta).
\end{aligned}
$$

Theorem 2.7 (Multiplication) *Let $x(\mathbf{t})$ and $y(\mathbf{t})$ in $\mathbb{L}^2(\mathbb{R}^n)$, then*

$$
\mathcal{F}\left[x(\mathbf{t})y(\mathbf{t})\right](\zeta) = \frac{1}{\left(\sqrt{2\pi}\right)^n}\mathcal{F}\left[x(\mathbf{t})\right](\zeta)*\mathcal{F}\left[y(\mathbf{t})\right](\zeta).
$$

Proof. Let $s(\mathbf{t}) = x(\mathbf{t})y(\mathbf{t})$, then

$$\begin{aligned}
\mathcal{F}\left[s(\mathbf{t})\right](\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} x(\mathbf{t})y(\mathbf{t})e^{-i\zeta.\mathbf{t}}d\mathbf{t} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} y(\mathbf{t})\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{i\xi.\mathbf{t}}\widehat{x}\left(\xi\right)d\xi\right)e^{-i\zeta.\mathbf{t}}d\mathbf{t} \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \widehat{x}\left(\xi\right)\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} y(\mathbf{t})e^{-i(\zeta-\xi).\mathbf{t}}e^{-i.\mathbf{t}}d\mathbf{t}\right)d\xi \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \widehat{x}\left(\xi\right)\widehat{y}\left(\zeta-\xi\right)d\xi \\
&= \frac{1}{\left(\sqrt{2\pi}\right)^n}\mathcal{F}\left[x(\mathbf{t})\right](\zeta) * \mathcal{F}\left[y(\mathbf{t})\right](\zeta).
\end{aligned}$$

Theorem 2.8 (Linear transformation of the domain $\mathbb{R}^n$) *Let $x(\mathbf{t}) \in \mathbb{L}^2\left(\mathbb{R}^n\right)$, then*

$$\mathcal{F}\left[x(\mathbf{A}\mathbf{t})\right](\zeta) = \frac{1}{\left|\det\left(\mathbf{A}\right)\right|}\mathcal{F}\left[x(\mathbf{t})\right]\left(\mathbf{A}^{-\top}\zeta\right),$$

where $\mathbf{A}$ is a nonsingular $n \times n$ matrix and $\mathbf{A}^{-T}$ denotes $\left(\mathbf{A}^{\top}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\top}$.

Proof. If we denote $y\left(\mathbf{t}\right) = x(\mathbf{A}\mathbf{t})$, then

$$\mathcal{F}\left[y(\mathbf{t})\right](\zeta) = \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}x(\mathbf{A}\mathbf{t})d\mathbf{t}$$

with the change of variables $\mathbf{s} = \mathbf{A}\mathbf{t}$, the Jacobian is $\frac{\partial(\mathbf{s})}{\partial(\mathbf{t})} = \det\left(\mathbf{A}\right)$, and using standard techniques for change of variables in an integral, we obtain

$$\begin{aligned}
\mathcal{F}\left[y(\mathbf{t})\right](\zeta) &= \frac{1}{\left|\det\left(\mathbf{A}\right)\right|\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\left(\mathbf{A}^{-1}\mathbf{s}\right)}x(\mathbf{s})d\mathbf{s} \\
&= \frac{1}{\left|\det\left(\mathbf{A}\right)\right|\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\left(\mathbf{A}^{-\top}\zeta\right).\mathbf{s}}x(\mathbf{s})d\mathbf{s} \\
&= \frac{1}{\left|\det\left(\mathbf{A}\right)\right|}\mathcal{F}\left[x(\mathbf{t})\right]\left(\mathbf{A}^{-\top}\zeta\right)
\end{aligned}$$

where we use the identity $\mathbf{a}.\left(\mathbf{C}\mathbf{b}\right) = \left(\mathbf{C}^{\top}\mathbf{a}\right).\mathbf{b}$ or equivalently $\mathbf{a}^{\top}.\mathbf{C}\mathbf{b} = \left(\mathbf{C}^{\top}\mathbf{a}\right)^{\top}\mathbf{b}$.

Theorem 2.9 (Differentiation) *Let $x(\mathbf{t}) \in \mathbb{L}^2\left(\mathbb{R}^n\right)$, then*

$$\mathcal{F}\left[\nabla_{\mathbf{t}}x(\mathbf{t})\right](\zeta) = i\zeta\mathcal{F}\left[x(\mathbf{t})\right](\zeta).$$

Proof. We define the gradient vector $\mathbf{y}(\mathbf{t}) = \nabla_{\mathbf{t}}x(\mathbf{t}) = \left[\frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, ..., \frac{\partial x}{\partial tn}\right]^{\top}$, taking the derivative of synthesis equation for x to get the synthesis equation for y_i

$$y_i(\mathbf{t}) = \frac{\partial x}{\partial t_i}(\mathbf{t}) = \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \widehat{x}(\mathbf{t})i\zeta_i e^{i\zeta.\mathbf{t}}d\zeta$$

Thus $\mathcal{F}\left[y_i(\mathbf{t})\right](\zeta) = i\zeta_i\mathcal{F}\left[x(\mathbf{t})\right](\zeta)$ and in matrix form $\mathcal{F}\left[\mathbf{y}(\mathbf{t})\right](\zeta) = i\zeta\mathcal{F}\left[x(\mathbf{t})\right](\zeta)$.

Theorem 2.10 (Differentiation in frequency) *Let* $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$, *then*

$$\mathcal{F}[\mathbf{t}x(\mathbf{t})](\zeta) = i\nabla_\zeta \mathcal{F}[x(\mathbf{t})](\zeta).$$

Proof. Let $\mathbf{y}(\mathbf{t}) = \mathbf{t}x(\mathbf{t})$, taking the derivative of analysis equation for x with respect to ζ_i

$$\begin{aligned}\frac{\partial \mathcal{F}[x(\mathbf{t})]}{\partial \zeta_i}(\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} x(\mathbf{t})(-it_i)\, e^{-i\zeta.\mathbf{t}} d\mathbf{t} \\ &= -i\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} y_i(\mathbf{t}) e^{-i\zeta.\mathbf{t}} d\mathbf{t} \\ &= -i\mathcal{F}[y_i(\mathbf{t})](\zeta)\end{aligned}$$

Thus $\mathcal{F}[\mathbf{y}(\mathbf{t})](\zeta) = i\nabla_\zeta \mathcal{F}[x(\mathbf{t})](\zeta)$.

Theorem 2.11 (Complex conjugation) *Let* $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$, *then*

$$\mathcal{F}\left[\overline{x(\mathbf{t})}\right](\zeta) = \overline{\mathcal{F}[x(\mathbf{t})](-\zeta)}.$$

Proof. Let $y(\mathbf{t}) = \overline{x(\mathbf{t})}$, then

$$\begin{aligned}\mathcal{F}\left[\overline{x(\mathbf{t})}\right](\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \overline{x(\mathbf{t})} e^{-i\zeta.\mathbf{t}} d\mathbf{t} \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \overline{x(\mathbf{t}) e^{i\zeta.\mathbf{t}}} d\mathbf{t} \\ &= \overline{\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} x(\mathbf{t}) e^{i\zeta.\mathbf{t}} d\mathbf{t}} \\ &= \overline{\mathcal{F}[x(\mathbf{t})](-\zeta)}.\end{aligned}$$

Theorem 2.12 (Duality) *Let* $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$, *then*

$$\mathcal{F}[\widehat{x}(\mathbf{t})](\zeta) = x(-\zeta).$$

Proof. Let $y(\mathbf{t}) = \widehat{x}(\mathbf{t})$, then

$$\begin{aligned}\mathcal{F}[y(\mathbf{t})](\zeta) &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \widehat{x}(\mathbf{t}) e^{-i\zeta.\mathbf{t}} d\mathbf{t} \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} \widehat{x}(\mathbf{t}) e^{i(-\zeta).\mathbf{t}} d\mathbf{t} \\ &= x(-\zeta).\end{aligned}$$

Theorem 2.13 (Separability) *Let* $x_1(t_1), ..., x_n(t_n) \in \mathbb{L}^2(\mathbb{R})$, *then*

$$\mathcal{F}[x_1(t_1)...x_n(t_n)](\zeta_1, ..., \zeta_n) = \mathcal{F}[x_1(t_1)](\zeta_1)...\mathcal{F}[x_n(t_n)](\zeta_n).$$

Proof. This follow from the separability of the complex exponential

$$\begin{aligned}&\frac{1}{\sqrt{(2\pi)}}\int_{\mathbb{R}}...\frac{1}{\sqrt{(2\pi)}}\int_{\mathbb{R}} x_1(t_1)...x_n(t_n) e^{-i(\zeta_1 t_1 + \zeta_2 t_2 + ... + \zeta_n t_n)} dt_1...dt_n \\ &= \left(\frac{1}{\sqrt{(2\pi)}}\int_{\mathbb{R}} x_1(t_1) e^{-i\zeta_1 t_1} dt_1\right)...\left(\frac{1}{\sqrt{(2\pi)}}\int_{\mathbb{R}} x_n(t_n) e^{-i\zeta_n t_n} dt_n\right) \\ &= \mathcal{F}[x_1(t_1)](\zeta_1)...\mathcal{F}[x_n(t_n)](\zeta_n).\end{aligned}$$

Theorem 2.14 (Parseval relation) *Let* $x(\mathbf{t})$ *and* $y(\mathbf{t})$ *in* $\mathbb{L}^2(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} x(\mathbf{t})\overline{y(\mathbf{t})}d\mathbf{t} = \int_{\mathbb{R}^n} \left(\mathcal{F}\left[x(\mathbf{t})\right](\zeta)\right)\left(\overline{\mathcal{F}\left[y(\mathbf{t})\right](\zeta)}\right)d\zeta.$$

Proof. Let $r(\mathbf{t}) = \overline{y(\mathbf{t})}$ and $s(\mathbf{t}) = x(\mathbf{t})r(\mathbf{t})$, then

$$\begin{aligned}\mathcal{F}\left[s(\mathbf{t})\right](\zeta) &= \int_{\mathbb{R}^n} \left(\mathcal{F}\left[x(\mathbf{t})\right](\xi)\right)\left(\mathcal{F}\left[r(\mathbf{t})\right](\zeta-\xi)\right)d\xi \\ &= \int_{\mathbb{R}^n} \left(\mathcal{F}\left[x(\mathbf{t})\right](\xi)\right)\left(\overline{\mathcal{F}\left[y(\mathbf{t})\right](\zeta-\xi)}\right)d\xi\end{aligned}$$

Evaluating at $\zeta = \mathbf{0}$,

$$\mathcal{F}\left[s(\mathbf{t})\right](\mathbf{0}) = \int_{\mathbb{R}^n} x(\mathbf{t})\overline{y(\mathbf{t})}d\mathbf{t} = \int_{\mathbb{R}^n} \left(\mathcal{F}\left[x(\mathbf{t})\right](\xi)\right)\left(\overline{\mathcal{F}\left[y(\mathbf{t})\right](\xi)}\right)d\xi.$$

If $x(\mathbf{t}) = y(\mathbf{t})$, we obtain $\int_{\mathbb{R}^n} |x(\mathbf{t})|^2 d\mathbf{t} = \int_{\mathbb{R}^n} |\mathcal{F}\left[x(\mathbf{t})\right](\xi)|^2 d\xi$.

3 Multidimensional windowed Fourier transform

One of the basic problems encountered in signal representations using conventional Fourier transform (FT) is the ineffectiveness of the Fourier kernel to represent and compute location information. One method to overcome such a problem is the windowed Fourier transform (WFT). Moreover, in practice, most natural signals are non-stationary. In order to characterize a non-stationary signal properly, the windowed Fourier transform (WFT) is commonly used. in this section, we introduce the multidimensional windowed Fourier transform.

Definition 2 *Let* Ψ *be a given multidimensional window function in* $\mathbb{L}^2(\mathbb{R}^n)$, *then the multidimensional window Fourier transform (MWFT) of any function* $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$ *is defined and denoted as*

$$\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta) = \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}}x(\mathbf{t})\overline{\Psi\left(\mathbf{t}-\mathbf{b}\right)}d\mathbf{t}, \qquad \mathbf{b},\zeta \in\mathbb{R}^n. \tag{3.1}$$

Further, the WFT (3.1) can be rewritten as

$$\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta) = \mathcal{F}\left[x(\mathbf{t})\overline{\Psi\left(\mathbf{t}-\mathbf{b}\right)}\right], \tag{3.2}$$

Applying inverse FT (2.2), (3.2) yields

$$\begin{aligned}x(\mathbf{t})\overline{\Psi\left(\mathbf{t}-\mathbf{b}\right)} &= \mathcal{F}^{-1}\left[\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta)\right] \\ &= \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}}\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta)\,d\zeta\end{aligned} \tag{3.3}$$

Multiplying (3.3) both sides by $\Psi\left(\mathbf{t}-\mathbf{b}\right)$ and then integrating with respect to $d\mathbf{b}$, we get

$$x(\mathbf{t})\left\|\Psi\right\|^2 = \frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}}\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta)\,\Psi\left(\mathbf{t}-\mathbf{b}\right)d\zeta d\mathbf{b}.$$

Equivalently, we have

$$x(\mathbf{t}) = \frac{1}{\left(\sqrt{2\pi}\right)^n\left\|\Psi\right\|^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}}\mathcal{V}_\Psi\left[x(\mathbf{t})\right](\mathbf{b},\zeta)\,\Psi\left(\mathbf{t}-\mathbf{b}\right)d\zeta d\mathbf{b}. \tag{3.4}$$

equation (3.4) gives the inversion formula corresponding to MWFT (3.1).

Theorem 3.1 *For any two functions $x(\mathbf{t})$ and $y(\mathbf{t})$ in $\mathbb{L}^2(\mathbb{R}^n)$, we have following*

i) *Linearity*

$$\mathcal{V}_\Psi [\lambda x(\mathbf{t}) + \mu y(\mathbf{t})](\mathbf{b},\zeta) = \lambda \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta) + \mu \mathcal{V}_\Psi [y(\mathbf{t})](\mathbf{b},\zeta).$$

ii) *Orthogonality*

$$\langle \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta), \mathcal{V}_\Psi [y(\mathbf{t})](\mathbf{b},\zeta)\rangle = \|\Psi\|^2 \langle x(\mathbf{t}), y(\mathbf{t})\rangle.$$

Proof. For ii), by definition (2), we have

$$\begin{aligned}
\langle \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta), \mathcal{V}_\Psi [y(\mathbf{t})](\mathbf{b},\zeta)\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta)\, \overline{\mathcal{V}_\Psi [y(\mathbf{t})](\mathbf{b},\zeta)}\, d\zeta d\mathbf{b} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta)\, \overline{\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}} y(\mathbf{t})\, \overline{\Psi(\mathbf{t}-\mathbf{b})} d\mathbf{t}\right)}\, d\zeta d\mathbf{b} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}} \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta)\, d\zeta\right) \overline{y(\mathbf{t})}\Psi(\mathbf{t}-\mathbf{b})\, d\mathbf{t} d\mathbf{b}.
\end{aligned}$$

By virtue of Equation (3.3), we have

$$\begin{aligned}
\langle \mathcal{V}_\Psi [x(\mathbf{t})](\mathbf{b},\zeta), \mathcal{V}_\Psi [y(\mathbf{t})](\mathbf{b},\zeta)\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x(\mathbf{t})\overline{\Psi(\mathbf{t}-\mathbf{b})}\Psi(\mathbf{t}-\mathbf{b})\,\overline{y(\mathbf{t})} d\mathbf{t} d\mathbf{b} \\
&= \int_{\mathbb{R}^n} x(\mathbf{t})\overline{y(\mathbf{t})} d\mathbf{t} \int_{\mathbb{R}^n} \overline{\Psi(\mathbf{t}-\mathbf{b})}\Psi(\mathbf{t}-\mathbf{b})\, d\mathbf{b} \\
&= \|\Psi\|^2 \langle x(\mathbf{t}), y(\mathbf{t})\rangle.
\end{aligned}$$

Next, we introduce the multidimensional fractional Fourier transform as a generalization of the classical multidimensional Fourier transform.

4 Multidimensional fractional Fourier transform

The fractional Fourier transform was introduced by Namias (1980) and a rigorous mathematical framework of the properties of fractional Fourier transform the Schwartz space of rapidly decreasing functions was given by McBride and Kerr (1987). Let us define multidimensional fractional Fourier transform.

Definition 3 *Let $x(\mathbf{t})$ be a signal in $\mathbb{L}^2(\mathbb{R}^n)$. The multidimensional fractional Fourier transform with order $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in (-\pi,\pi)^n$ on $\mathbb{L}^1(\mathbb{R}^n)$ of $x(\mathbf{t})$ is defined by*

$$\mathcal{F}_\alpha [x(\mathbf{t})](\zeta) = \int_{\mathbb{R}^n} K_\alpha(\mathbf{t},\zeta)\, x(\mathbf{t})\, d\mathbf{t}, \tag{4.1}$$

where $K_\alpha(\mathbf{t},\zeta) = \prod_{i=1}^n K_{\alpha_i}(t_i,\zeta_i)$ and $K_{\alpha_i}(t_i,\zeta_i)$ the kernel of the FRFT and is given by

$$K_{\alpha_i}(t_i,\zeta_i) = \begin{cases} \frac{c(\alpha_i)}{\sqrt{2\pi}} e^{i\left\{a(\alpha_i)\left[t_i^2+\zeta_i^2-2b(\alpha_i)t_i\zeta_i\right]\right\}} & \text{for } \alpha_i \neq k\pi, \\ \delta(t_i-\zeta_i) & \text{for } \alpha_i = 2k\pi, \\ \delta(t_i+\zeta_i) & \text{for } \alpha_i = (2k\pm1)\pi, k\in\mathbb{Z} \end{cases}$$

where $a(\alpha_i) = \cot\alpha_i/2, b(\alpha_i) = \sec\alpha_i, c(\alpha_i) = \sqrt{1 - i\cot\alpha_i}$, *and the corresponding inversion formula is also a MFRFT is given by*

$$x(\mathbf{t}) = \mathcal{F}_{-\alpha}\left\{\mathcal{F}_{\alpha}\left[x(\mathbf{t})\right](\zeta)\right\}(\mathbf{t}) = \int_{\mathbb{R}^n} \mathcal{F}_{\alpha}\left[x(\mathbf{t})\right](\zeta) K_{-\alpha}(\mathbf{t}, \zeta) \, d\zeta. \tag{4.2}$$

It is easy to see that, when $\alpha_i = 0, \pi/2, \pi$ and and $3\pi/2$ for $i = 1, ..., n$, the MFRFT is reduced to the identity operation, the MFT, time-reverse operation, and the MIFT, respectively. For each $\lambda \in \mathbb{R}\setminus\{0\}$, we define

$$e_{\alpha,\lambda}(\mathbf{t}) = e^{i\lambda\sum_{i=1}^{n} a(\alpha_i)t_i^2}, \forall \mathbf{t} \in \mathbb{R}^n.$$

It is easy to observe that $e_{\alpha,-\lambda}(\mathbf{t}) = e_{-\alpha,\lambda}(\mathbf{t})$ and the multidimensional fractional Fourier transform of $x(\mathbf{t})$ (4.1) can be rewritten as

$$\mathcal{F}_{\alpha}\left[x(\mathbf{t})\right](\zeta) = c(\alpha)\, e_{\alpha,1}(\zeta)\, \mathcal{F}(e_{\alpha,1}x)(\zeta_1 \csc\alpha_1, ..., \zeta_n \csc\alpha_n), \tag{4.3}$$

where $c(\alpha) = c(\alpha_1) ....c(\alpha_n)$ and $\mathcal{F}(e_{\alpha,1}x)$ is the Fourier transform of $e_{\alpha,1}x$. Using this notation, we can rewrite

$$K_{\alpha}(\mathbf{t}, \zeta) = \frac{c(\alpha)}{\left(\sqrt{2\pi}\right)^n} e_{\alpha,1}(\mathbf{t})\, e_{\alpha,1}(\zeta)\, e^{-i\sum_{i=1}^{n} t_i\zeta_i \csc\alpha_i}.$$

From (4.3), it is clear that $\mathcal{F}_{\alpha}\left[x(\mathbf{t})\right](\zeta) \in C_0(\mathbb{R}^n)$ for all $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$.
Next, we highlight some properties of MFRFT.

Theorem 4.1 *Let* $x(\mathbf{t})$, $y(\mathbf{t})$ *in* $\mathbb{L}^2(\mathbb{R}^n)$ *and* $\mathbf{k}, \zeta_0 \in \mathbb{R}^n$, *then the MFRFT satisfies the following orthogonality Relation:*

$$\left\langle \mathcal{F}_{\alpha}\left[x(\mathbf{t})\right], \mathcal{F}_{\alpha}\left[y(\mathbf{t})\right]\right\rangle = \left\langle x(\mathbf{t}), y(\mathbf{t})\right\rangle.$$

Proof. We have

$$\begin{aligned}
\left\langle \mathcal{F}_{\alpha}\left[x(\mathbf{t})\right], \mathcal{F}_{\alpha}\left[y(\mathbf{t})\right]\right\rangle &= \int_{\mathbb{R}^n} \mathcal{F}_{\alpha}\left[x(\mathbf{t})\right](\zeta)\, \overline{\mathcal{F}_{\alpha}\left[y(\mathbf{t})\right](\zeta)} d\zeta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{\alpha}(\mathbf{t}, \zeta)\, x(\mathbf{t})\, \overline{K_{\alpha}(\mathbf{s}, \zeta)\, y(\mathbf{s})\, d\mathbf{s}} d\mathbf{t} d\zeta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x(\mathbf{t})\, \overline{y(\mathbf{s})} \left(\int_{\mathbb{R}^n} K_{\alpha}(\mathbf{t}, \zeta)\, \overline{K_{\alpha}(\mathbf{s}, \zeta)} d\zeta\right) d\mathbf{s} d\mathbf{t} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} x(\mathbf{t})\, \overline{y(\mathbf{s})} \delta(\mathbf{t} - \mathbf{s})\, d\mathbf{s} d\mathbf{t} \\
&= \int_{\mathbb{R}^n} x(\mathbf{t})\, \overline{y(\mathbf{s})} d\mathbf{t} \\
&= \left\langle x(\mathbf{t}), y(\mathbf{t})\right\rangle.
\end{aligned}$$

In the following, we introduce multidimensional linear canonical transform, which is a generalized version of the classical Fourier transform with four parameters.

5 Multidimensional linear canonical transform

The linear canonical transform (LCT) introduced by Moshinsky and Quesne (1971) have proved useful and appropriate for investigating deep problems in science and engineering (Bhat 2023). It encompasses several well known signal processing transforms as special cases including the Fourier transform, the fractional Fourier transform, the Fresnel transform, and even simple multiplication by quadratic phase factors (Healy et al. 2016). Despite enormous lucubrations in the theory of linear canonical transforms, the multidimensional LCT involving a general $2n \times 2n$ real, symplectic matrix M with $n(2n+1)$ independent parameters is yet to be explored exclusively. We shall define multidimensional linear canonical transform (MLCT).

Definition 4 *For any $2n \times 2n$ real, symplectic matrix $M_{2n\times 2n} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the multidimensional linear canonical transform for any $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$ is defined by*

$$\mathcal{L}_M[x(\mathbf{t})](\zeta) = \int_{\mathbb{R}^n} x(\mathbf{t})\,\mathcal{K}_M(\mathbf{t},\zeta)\,d\mathbf{t} \tag{5.1}$$

where $\mathcal{K}_M(\mathbf{t},\zeta)$ is the kernel and is given by

$$\mathcal{K}_M(\mathbf{t},\zeta) = \Omega(B,n)\exp\left\{ i\frac{\left(\zeta^\top DB^{-1}\zeta - 2\zeta^\top B^{-\top}\mathbf{t} + \mathbf{t}^\top B^{-1}A\mathbf{t}\right)}{2}\right\}, \qquad \det|B| \neq 0 \tag{5.2}$$

where $\Omega(B,n) = \frac{1}{(2\pi)^{n/2}|\det B|^{1/2}}$.

For a given real, symplectic matrix M, the multidimensional linear canonical transform kernel (5.2) satisfies the following properties:

i $\mathcal{K}_{M^{-1}}(\zeta,\mathbf{t}) = \overline{\mathcal{K}_M(\mathbf{t},\zeta)}$,

ii) $\int_{\mathbb{R}^n}\mathcal{K}_M(\mathbf{t},\zeta)\,\mathcal{K}_{M^{-1}}(\omega,\mathbf{t})\,d\mathbf{t} = \delta(\omega-\zeta)$.

The inversion formula associated with the multidimensional LCT is given by

$$f(\mathbf{t}) = \mathcal{L}_{M^{-1}}\{\mathcal{L}_M[x(\mathbf{t})](\zeta)\}(\mathbf{t}) = \int_{\mathbb{R}^n}\mathcal{L}_M[x(\mathbf{t})](\zeta)\,\overline{\mathcal{K}_M(\mathbf{t},\zeta)}d\zeta.$$

For typographical convenience we write the $2n \times 2n$ matrix $M = (A, B : C, D)$. the LCT boils down to various integral transforms such as:

i) When the sub-matrices of the real, symplectic matrix $M = (A, B : C, D)$ are chosen as $A = diag(a_{11}, ..., a_{nn})$, $B = (b_{11}, ..., b_{nn})$, $C = diag(c_{11}, ..., c_{nn})$ and , $D = (d_{11}, ..., d_{nn})$, the multidimensional LCT (5.1) yields the n-dimensional separable linear canonical transform:

$$\mathcal{L}_M[x(\mathbf{t})] = \frac{1}{(2\pi)^{n/2}\left|\prod_{i=1}^n b_{ii}\right|^{1/2}}\int_{\mathbb{R}^n} x(\mathbf{t})\exp\left\{ i\frac{\left(d_{ii}\zeta_i^2 - 2\zeta_i t_i + a_{ii}t_i^2\right)}{2b_{ii}}\right\} d\mathbf{t}.$$

ii) For the symplectic matrix $M = (I_n\cos\theta, I_n\sin\theta, -I_n\sin\theta, I_n\cos\theta)$, the multidimensional LCT (5.1) yields the n-dimensional non-separable fractional Fourier transform:

$$\mathcal{F}^\theta[x(\mathbf{t})](\zeta) = \frac{1}{(2\pi)^{n/2}|\sin\theta|^{n/2}}\int_{\mathbb{R}^n} x(\mathbf{t})\exp\left\{\frac{i\left(\zeta^\top\zeta + \mathbf{t}^\top\mathbf{t}\right)\cot\theta}{2} - i\zeta^\top\mathbf{t}\csc\theta\right\} d\mathbf{t}.$$

iii) In the case the sub-matrices the real, sympletic matrix $M=(A,B:C,D)$ are chosen as $A=D=diag\,(\cos\theta_1,...,\cos\theta_n)\,,$ $B=-C=(\sin\theta_1,...,\sin\theta_n)\,,$ the multidimensional LCT (5.1) yields the n-dimensional separable fractional Fourier transform:

$$\mathcal{F}^{(\theta_1,...,\theta_n)}\left[x\left(\mathbf{t}\right)\right](\zeta)=\frac{1}{(2\pi)^{n/2}\left|\prod_{i=1}^{n}\sin\theta_i\right|^{1/2}}\int_{\mathbb{R}^n}x\left(\mathbf{t}\right)\exp\left\{i\sum_{i=1}^{n}\left(\frac{\left(\zeta_i^2+t_i^2\right)\cot\theta_i}{2}-\zeta_i t_i\csc\theta_i\right)\right\}d\mathbf{t}.$$

iv) For the matrix $M=(I_n,B:\mathbf{0},I_n)\,,$ the multidimensional LCT (5.1) reduces to the n-dimensional non-separable Fresnel transform:

$$\mathcal{F}_M\left[x\left(\mathbf{t}\right)\right](\zeta)=\frac{1}{(2\pi)^{n/2}\left|\det B\right|^{1/2}}\int_{\mathbb{R}^n}x\left(\mathbf{t}\right)\exp\left\{i\frac{\left(\zeta^\top B^{-1}\zeta-2\zeta^\top B^{-\top}\mathbf{t}+\mathbf{t}^\top B^{-1}\mathbf{t}\right)}{2}\right\}d\mathbf{t}.$$

v) Choosing the sub-matrices as $A=D=I_n, B=diag\,(b_{11},...,b_{nn})\,, C=\mathbf{0}$ the multidimensional LCT (5.1) reduces to the n-dimensional non-separable Fresnel transform:

$$\mathcal{F}_{(b_{11},...,b_{nn})}\left[x\left(\mathbf{t}\right)\right](\zeta)=\frac{1}{(2\pi)^{n/2}\left|\prod_{i=1}^{n}b_{ii}\right|^{1/2}}\int_{\mathbb{R}^n}x\left(\mathbf{t}\right)\exp\left\{i\sum_{i=1}^{n}\left(\frac{\left(\zeta_i^2-2\zeta_i t_i+t_i^2\right)}{2b_{ii}}\right)\right\}d\mathbf{t}.$$

vi) When $M=(\mathbf{0},I_n:-I_n,\mathbf{0})$, the multidimensional LCT (5.1) reduces to the classical n-dimensional Fourier transform:

$$\mathcal{F}\left[x\left(\mathbf{t}\right)\right](\zeta)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}x\left(\mathbf{t}\right)e^{-i\zeta.\mathbf{t}}d\mathbf{t}.$$

Here, we shall digress a bit to gain an intuition regarding the computational complexity of the multi-dimensional linear canonical transform defined in (5.1). For any $x\left(\mathbf{t}\right)\in\mathbb{L}^2\left(\mathbb{R}^n\right)$, the multi-dimensional LCT with respect to a real, symplectic matrix $M=(A,B:C,D)$ can be expressed as

$$\begin{aligned}\mathcal{L}_M\left[x\left(\mathbf{t}\right)\right](\zeta)&=\frac{1}{(2\pi)^{n/2}\left|\det B\right|^{1/2}}\exp\left\{i\frac{\zeta^\top DB^{-1}\zeta}{2}\right\}\int_{\mathbb{R}^n}x\left(\mathbf{t}\right)\exp\left\{\frac{i\left(\mathbf{t}^\top B^{-1}A\mathbf{t}\right)}{2}\right\}e^{-i\left(B^{-1}\zeta\right)^\top\mathbf{t}}d\mathbf{t} \qquad (5.3)\\&=\frac{1}{(2\pi)^{n/2}\left|\det B\right|^{1/2}}\exp\left\{i\frac{\zeta^\top DB^{-1}\zeta}{2}\right\}\mathcal{F}\left[g\left(\mathbf{t}\right)\right]\left(B^{-1}\zeta\right)\end{aligned}$$

where $g(\mathbf{t})=x(\mathbf{t})\exp\left\{\frac{i\left(\mathbf{t}^\top B^{-1}A\mathbf{t}\right)}{2}\right\}$.

Thus, it is clear from (5.3), that multidimensional LCT can be regarded as a chirp-Fourier-chirp transformation. Therefore, the computational complexity of the multi-dimensional linear canonical transform is determined by that of the traditional Fourier transform. As such, the conventional fast Fourier transform can be employed for executing a speedy computation of the multi-dimensional linear canonical transform.

Next, we investigate some basic properties associated with LCT.

Theorem 5.1 *Let $x(\mathbf{t}),y(\mathbf{t})\in\mathbb{L}^2\left(\mathbb{R}^n\right)$ and $\mathbf{k}$,$\zeta_0\in\mathbb{R}^n$, then the MLCT satisfies following properties:*

1. *Parity:*

$$\mathcal{L}_M\left[x(-\mathbf{t})\right](\zeta)=\mathcal{L}_M\left[x(\mathbf{t})\right](-\zeta)\,.$$

2. *Orthogonality Relation:*

$$\langle \mathcal{L}_M [x(\mathbf{t})], \mathcal{L}_M [y(\mathbf{t})] \rangle = \langle x(\mathbf{t}), y(\mathbf{t}) \rangle .$$

Proof. To be specific, we shall only prove the parity property, the rest of the propertie follows similarly. For any $\mathbf{k} \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathcal{L}_M [x(-\mathbf{t})](\zeta) &= \int_{\mathbb{R}^n} \mathcal{K}_M(\mathbf{t}, \zeta)\, x(-\mathbf{t})\, d\mathbf{t} \\
&= \frac{1}{(2\pi)^{n/2} |\det B|^{1/2}} \int_{\mathbb{R}^n} x(\mathbf{t}) \exp\left\{ i \frac{\left(\zeta^\top D B^{-1} \zeta + 2\zeta^\top B^{-\top} \mathbf{t} + \mathbf{t}^\top B^{-1} A \mathbf{t}\right)}{2} \right\} d\mathbf{t} \\
&= \mathcal{L}_M [x(\mathbf{t})](-\zeta) .
\end{aligned}$$

6 Multidimensional quadratic-phase Fourier transform

The most neoteric generalization of the classical Fourier transform (FT) with five real parameters appeared via the theory of reproducing kernels is known as the quadratic-phase Fourier transform (QPFT). It treats both the stationary and nonstationary signals in a simple and insightful way that are involved in radar, signal processing, and other communication systems (see Bhat 2023 and references therein). Here, we gave the notation and definition of the quadratic-phase Fourier transform and study some of its properties.

In this section we introduce the definition of the multidimensional quadratic-phase Fourier transform which is a generalization of the classical quadratic-phase Fourier transform.

Definition 5 *For a real parameter set $\Lambda = (a, b, c, d, e)$ with $b \neq 0$, the multidimensional quadratic-phase Fourier transform of any signal $x(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$ is defined as*

$$Q_\Lambda [x(\mathbf{t})](\zeta) = \int_{\mathbb{R}^n} K_\Lambda(\mathbf{t}, \zeta)\, x(\mathbf{t})\, d\mathbf{t}, \tag{6.1}$$

where $K_\alpha(\mathbf{t}, \zeta) = \prod_{i=1}^{n} K_{\alpha_i}(t_i, \zeta_i)$ and $K_{\alpha_i}(t_i, \zeta_i)$ the kernel of the quadratic-phase Fourier transform and is given (QFFT) and is given by

$$K_{\alpha_i}(t_i, \zeta_i) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -i \left(a t_i^2 + b t_i \zeta_i + c \zeta_i^2 + d t_i + e \zeta_i \right) \right\} \tag{6.2}$$

and corresponding inversion formula is given by

$$x(\mathbf{t}) = Q_\Lambda^{-1} \left(Q_\Lambda [x(\mathbf{t})](\zeta) \right)(\mathbf{t}) = \int_{\mathbb{R}^n} \overline{K_\Lambda(\mathbf{t}, \zeta)}\, Q_\Lambda [x(\mathbf{t})](\zeta)\, d\zeta.$$

By appropriately choosing parameters in $\Lambda = (a, b, c, d, e)$, definition (5) allows us to make the following comments regarding the notion of multidimensional quadratic-phase Fourier transform:

i) Choosing the parametric set $\Lambda = (0, 1, 0, 0, 0)$, the MQPFT (6.1) boils down to the classical multidimensional Fourier transform (FT):

$$Q_\Lambda [x(\mathbf{t})](\zeta) = \mathcal{F}[x(\mathbf{t})](\zeta) .$$

ii) When $\Lambda = (-\cot\theta_i/2, \csc\theta_i, -\cot\theta_i/2, 0, 0)$, $i = 1, ..., n$ then, multiplying (6.1) with $\sqrt{1 - i\cot\theta_i}, i = 1, ..., n$ yields the multidimensional fractional Fourier transform

$$Q_\Lambda[x(\mathbf{t})](\zeta) = \mathcal{F}_\theta[x(\mathbf{t})](\zeta).$$

iii) When $\Lambda = \left(-\frac{a}{2b}, \frac{1}{b}, -\frac{c}{2b}, 0, 0\right)$, and then multiplying (6.1) with $1/\sqrt{ib}$ the MQPFT (6.1) becomes the MlCT:

$$Q_\Lambda[x(\mathbf{t})](\zeta) = \mathcal{L}_\Lambda[x(\mathbf{t})](\zeta).$$

Now, we will establish some properties of the multidimensional quadratic-phase Fourier transform.

Theorem 6.1 *Let $x(\mathbf{t}), y(\mathbf{t}) \in \mathbb{L}^2(\mathbb{R}^n)$ and $\mathbf{k}$,$\zeta_0 \in \mathbb{R}^n$, then the MQPFT satisfies following properties:*

1. *Linearity:*
$$Q_\Lambda[\alpha x(\mathbf{t}) + \beta y(\mathbf{t})](\zeta) = \alpha Q_\Lambda[x(\mathbf{t})](\zeta) + \beta Q_\lambda[y(\mathbf{t})].$$

2. *Parity:*
$$Q_\Lambda[x(-\mathbf{t})](\zeta) = Q_{\Lambda'}[x(\mathbf{t})](-\zeta),$$
where $\Lambda' = (a, b, c, -d, -e)$

3. *Conjugation:*
$$Q_\Lambda\left[\overline{x(\mathbf{t})}\right](\zeta) = \overline{Q_{-\Lambda}[x(\mathbf{t})](\zeta)},$$
where $-\Lambda = (-a, -b, -c, -d, -e)$

4. *Orthogonality Relation:*
$$\langle Q_\Lambda[x(\mathbf{t})], Q_M[y(\mathbf{t})]\rangle = \frac{1}{b^n}\langle x(\mathbf{t}), y(\mathbf{t})\rangle.$$

Proof. For the sake of brevity, we avoid proof.

7 Conclusion

In this paper a dynamically analysis was conducted to investigate possible extension of one dimensional transform to a generalized multidimensional one. The proposed tools, namely Fourier transform and multidimensional scaling, proved to be assertive methods to analyze such transform, the first is to generalize the dynamics and the second for revealing the clusters. In future, this approach should be applied for other transforms characteristics like the wavelets and/or Laplace. In this perspective, the replicated multidimensional technique can be used to analyze the spatial data or spatiotemporal economics. So, we have proposed the generalized multidimentional Fourier frequency inherently, associated with multidimensional Fourier transform. Therfore, we are able to provide physical meaning of so called negative frequencies in multidimensional Fourier transform $(M-FT)$, which in turn provide multidimensional spatial and spatio-temporal data analysis. The complex exponential representation of sinusoidal function always yields two frequencies, negative frequency corresponding to positive frequency and vice versa, in the multidimensional Fourier spectrum. Thus, using the $M-FT$, we propose multidimensional transform and associated multidimensional analytic signal $(M-S)$ with following properties: (a) the extra and redundant positive, negative, or both frequencies, introduced due to complex exponential representation of multidimensional Fourier spectrum, are suppressed, (b) real part of $M-S$ is original signal, (c) real and imaginary part of $M-S$ are orthogonal, and (d) the magnitude envelope of an original signal is obtained as the magnitude of its associated $M-S$, which is the instantaneous amplitude of the $M-S$. The proposed $M-T$ and associated $M-S$ are generalization respectively.

Appendix

A summary table of some **properties of Multidimensional Fourier transform**

Domain (non-periodic)	**Continuous-domain**	**Discrete-domain** (Λ)
Name of the transform	Continuous-domain Fourier transform	Discrete-domain Fourier transform
Signals and domains	$x(\mathbf{t}) \longleftrightarrow \mathcal{F}[x(\mathbf{t})](\zeta) = \widehat{x}(\mathbf{t}),$ $\mathbf{t}, \mathbf{k}, \zeta, \zeta_0 \in \mathbb{R}^n$	$x(\mathbf{t}) \longleftrightarrow \mathcal{F}[x(\mathbf{t})](\zeta) = \widehat{x}(\mathbf{t}),$ $\mathbf{t}, \mathbf{k} \in \Lambda, \zeta, \zeta_0 \in \mathbb{R}^n$
Periodicity	none	$\mathcal{F}(x[\mathbf{t}])(\zeta+\xi) = \mathcal{F}[x(\mathbf{t})](\zeta), \xi \in \overline{\Lambda}$
Period	none	$\mathbf{t}, \mathcal{P}_{\overline{\Lambda}}, \lvert \mathcal{P}_{\overline{\Lambda}} \rvert = 1/d(\Lambda)$
Analysis	$\mathcal{F}[x(\mathbf{t})](\zeta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\zeta.\mathbf{t}} x(\mathbf{t})\, d\mathbf{t}$	$\mathcal{F}(x[\mathbf{t}])(\zeta) = \sum_{\mathbf{t}\in\Lambda} e^{-i2\pi\zeta.\mathbf{t}} x[\mathbf{t}]$
Synthesis	$x(\mathbf{t}) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{i\zeta.\mathbf{t}} \mathcal{F}[x(\mathbf{t})](\zeta)\, d\zeta$	$x[\mathbf{t}] = d(\Lambda) \int_{\mathcal{P}_{\overline{\Lambda}}} e^{i2\pi\zeta.\mathbf{t}} \mathcal{F}(x[\mathbf{t}])(\zeta)\, d\zeta$
Linearity	$\mathcal{F}[Ax(\mathbf{t}) + By(\mathbf{t})](\zeta) = A\widehat{x}(\mathbf{t}) + B\widehat{y}(\mathbf{t})$	$\mathcal{F}[Ax[\mathbf{t}] + By[\mathbf{t}]](\zeta) = A\widehat{x}(\mathbf{t}) + B\widehat{y}(\mathbf{t})$
Translation	$\mathcal{F}[x(\mathbf{t}-\mathbf{k})](\zeta) = e^{-i\mathbf{k}.\zeta}\widehat{x}(\mathbf{t})$	$\mathcal{F}(x[\mathbf{t}-\mathbf{k}])(\zeta) = e^{i2\pi\mathbf{k}.\zeta}\widehat{x}(\mathbf{t})$
Modulation	$\mathcal{F}\left[e^{i\zeta_0.\mathbf{t}} x(\mathbf{t})\right](\zeta) = \mathcal{F}[x(\mathbf{t})](\zeta-\zeta_0)$	$\mathcal{F}\left(e^{i2\pi\zeta_0.\mathbf{t}} x[\mathbf{t}]\right)(\zeta) = \mathcal{F}(x[\mathbf{t}])(\zeta-\zeta_0),$
Convolution	$\mathcal{F}[(x*y)(\mathbf{t})](\zeta) = (2\pi)^{\frac{n}{2}}\, \widehat{x}(\mathbf{t})\widehat{y}(\mathbf{t})$	$\mathcal{F}((x*y)[\mathbf{t}])(\zeta) = \widehat{x}(\mathbf{t})\widehat{y}(\mathbf{t})$
Multiplication	$\mathcal{F}[x(\mathbf{t})y(\mathbf{t})](\zeta) = \frac{1}{(\sqrt{2\pi})^n}\widehat{x}(\mathbf{t}) * \widehat{y}(\mathbf{t})$	$\mathcal{F}[x[\mathbf{t}]\, y(\mathbf{t})](\zeta) = d(\Lambda)\, \widehat{x}(\mathbf{t}) * \widehat{y}(\mathbf{t})$
Automorphism of domain	$\mathcal{F}[x(\mathbf{At})](\zeta) = \frac{1}{\lvert \det(\mathbf{A}) \rvert}\mathcal{F}[x(\mathbf{t})]\left(\mathbf{A}^{-\top}\zeta\right)$	$\mathcal{F}[x(\mathbf{At})](\zeta) = \mathcal{F}(x[\mathbf{t}])\left(\mathbf{A}^{-\top}\zeta\right)$
Orthogonality	$\langle \mathcal{F}[x(\mathbf{t})], \mathcal{F}[y(\mathbf{t})] \rangle = \langle x(\mathbf{t}), y(\mathbf{t}) \rangle$	$\langle \mathcal{F}(x[\mathbf{t}]), \mathcal{F}(y[\mathbf{t}]) \rangle = \langle x(\mathbf{t}), y(\mathbf{t}) \rangle$
Differentiation	$\mathcal{F}[\nabla_{\mathbf{t}} x(\mathbf{t})](\zeta) = i\zeta\mathcal{F}[x(\mathbf{t})](\zeta)$	N/A
Differentiation in frequency	$\mathcal{F}[\mathbf{t}x(\mathbf{t})](\zeta) = i\nabla_\zeta \mathcal{F}[x(\mathbf{t})](\zeta)$	$\mathcal{F}(\mathbf{t}x[\mathbf{t}])(\zeta) = \frac{i}{(2\pi)^n}\nabla_\zeta \mathcal{F}(x[\mathbf{t}])(\zeta)$
Complex conjugation	$\mathcal{F}\left[\overline{x(\mathbf{t})}\right](\zeta) = \overline{\mathcal{F}[x(\mathbf{t})](-\zeta)}$	$\mathcal{F}\left(\overline{x[\mathbf{t}]}\right)(\zeta) = \overline{\mathcal{F}(x[\mathbf{t}])(-\zeta)}$
Parseval	$\int_{\mathbb{R}^n} x(\mathbf{t})\overline{y(\mathbf{t})}d\mathbf{t} = \int_{\mathbb{R}^n} (\widehat{x}(\mathbf{t}))\left(\overline{\widehat{y}(\mathbf{t})}\right) d\zeta$	$\sum_{\mathbf{t}\in\Lambda} x(\mathbf{t})\overline{y(\mathbf{t})}d\mathbf{t} = d(\Lambda)\int_{\mathcal{P}_{\overline{\Lambda}}} (\widehat{x}(\mathbf{t}))\left(\overline{\widehat{y}(\mathbf{t})}\right) d\zeta$
Duality	$\mathcal{F}[\widehat{x}(\mathbf{t})](\zeta) = x(-\zeta)$	$\mathcal{F}(\widetilde{x}(\mathbf{t}))[\zeta] = d(\Gamma)\, \widetilde{x}(-\zeta)$
Properties of Multidimensional Fourier transform		

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Conflicts of Interest

No conflict of interest is to be declared by any author.